Quantum Codes for Simplifying Design and Suppressing Decoherence in Superconducting Phase-Qubits

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We introduce simple qubit-encodings and logic gates which eliminate the need for certain difficult single-qubit operations in superconducting phase-qubits, while preserving universality. The simplest encoding uses two physical qubits per logical qubit. Two architectures for its implementation are proposed: one employing N physical qubits out of which $N/2$ are ancillas fixed in the $|1\rangle$ state, the other employing $N/2 + 1$ physical qubits, one of which is a bus qubit connected to all others. Details of a minimal set of universal encoded logic operations are given, together with recoupling schemes, that require nanosecond pulses. A generalization to codes with higher ratio of number of logical qubits per physical qubits is presented. Compatible decoherence and noise suppression strategies are also discussed.

I. INTRODUCTION

Solid state systems are now attracting much attention as potential components for quantum computers, in part because of their potential scalability, and because the parameters of a solid state qubit can be engineered with considerable freedom. This freedom allows one to add different building blocks to the quantum computer, each block bringing the design closer to satisfying the criteria required for quantum computation (QC) [1] and, in particular, enabling universal QC according to the (non fault-tolerant) "standard paradigm" [2]. In this standard paradigm, all single-qubit operations [generating $SU(2)$] plus an entangling two-qubit operation (e.g., controlled-NOT, denoted CNOT) are necessary to achieve universal quantum computation. Alternatively, one can use a discrete set of single-qubit operations (e.g., Hadamard and $\pi/8$) together with CNOT in order to approximate to arbitrary accuracy any quantum circuit [3], a method that is compatible with fault-tolerant quantum error correction [4, 5]. While the different building blocks help in reaching universality, each one comes with its fabrication difficulties and adds a potential source of noise and decoherence to the system.

It was realized recently that some of these building blocks can be replaced by 'software' means [6]. More specifically, by encoding a logical qubit into a few physical qubits, the design of the quantum computer can be simplified without compromising universality. This approach is known as encoded universality [7]. E.g., it is possible to remove difficult-to-implement single-qubit operations [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. This simplification has the advantages of facilitating fabrication and reducing some sources of decoherence and noise. Interesting alternatives to encoding that also aim to reduce design constraints, some by replacing logic gates with measurements, were recently presented in [19, 20, 21, 22].

Studies of simplifying encodings have so far been performed primarily for exchange-type Hamiltonians in spin-coupled solid state quantum computer designs [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23, 24, 25], and in NMR [26, 27]. Here we extend this study to superconducting phase qubits.

Superconducting phase qubits are among the leading solid state qubit candidates, in part due to recent experimental progress [28, 29]. Several designs of phase qubits have been suggested in the literature; see Ref. [30] for a review. In this work, we first focus on the d-wave grain boundary qubits [31, 32] and recall the relevant system Hamiltonian. A simplification of the design suggested in Refs. [31, 32] will yield a system Hamiltonian which is not versatile enough to be universal, according to the standard paradigm [2, 3], because it lacks certain single-qubit operations. We will show how universality can be recovered by using a simple encoding and recoupling techniques. The encoding will suggest the use of a "bus qubit", a concept that could be useful for other quantum computer designs. In particular, we explore the possibility of using these concepts with the other superconducting phase qubit designs. Finally, application of the dynamical decoupling technique to decoherence reduction will be examined.

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II. SYSTEM HAMILTONIAN

The d-wave grain boundary qubit (dGB qubit), Figure 1, has the following system Hamiltonian [32]:

$$H_S = H_X + H_Z + H_{ZZ}, \qquad (1)$$

where

$$H_X = \sum_{i=1}^{N} \Delta_i \sigma_i^x \quad \text{tunneling} \qquad (2)$$

$$H_Z = \sum_{i=1}^{N} b_i \sigma_i^z \quad \text{bias} \qquad (3)$$

$$H_{ZZ} = \sum_{i,j=1}^{N} J_{ij} \sigma_i^z \otimes \sigma_j^z \quad \text{Josephson coupling}, \qquad (4)$$

and where σ_i^α $(\alpha = x, y, z)$ are the Pauli matrices. In this system, coherent tunneling of the phase is only possible when the energy levels are in resonance [56]. As a result, when a bias or Josephson coupling is turned on, the tunneling matrix element(s) Δ_i for the corresponding qubit(s) is exponentially suppressed. We therefore take *only one* of the terms (2), (3) or (4) to be on at any given time.

Moreover, for the dGB qubit, turning on the bias or Josephson coupling is the only way to control the value of the tunneling matrix element. The latter can then effectively only be turned on or off, without continuous control over its magnitude. This magnitude is determined at fabrication time by the asymmetry between the d-wave superconductors forming the qubit and by the width of the junction [31]. It is interesting to note that by connecting an external capacitor to the qubit circuit, the magnitude of Δ_i could be controlled independently [33]. However, this design complicates fabrication and connects the qubit to another potential source of decoherence. Since the tunneling matrix elements depend exponentially on their parameters, small noise on these parameters can have dramatic effect on the coherence of the system. In this paper, we are interested in simplifying fabrication demands on the design and reducing sources of decoherence, hence this possibility will not be considered further.

We now come to our main simplification: *In this work we reduce the constraints on fabrication by removing the possibility of applying bias b_i on individual qubits.* This bias requires, e.g., the possibility of applying a local magnetic field on each qubit, and is experimentally very challenging to realize. We do retain the Josephson couplings J_{ij} between the qubits, as these are necessary to produce entanglement. These couplings are realized by connecting pairs of qubits by a superconducting single electron transistor (SSET) [31, 32]. The magnitude of J_{ij} can, to some extent, be tuned continuously by the SSET's gate
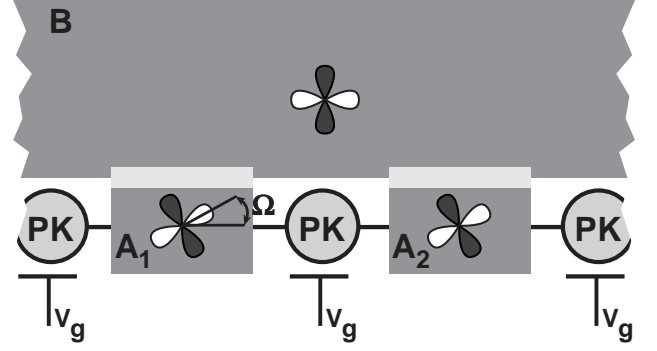

FIG. 1: Superconducting grain boundary qubits: Terminals A_1, A_2 and B are d-wave superconductors, PK (parity key) represent the SSETs, Ω is the mismatch angle between the lattices of A_i and B. Positive lobes of the d-wave order parameter are shaded. Two physical qubits are shown.

voltage. The sign of this energy could be changed by inserting a strong π-junction. Again, with simplification of design in mind, the sign of J_{ij} will hereafter be considered fixed.

The effective system Hamiltonian that we consider in this paper is therefore:

$$H_S = H_X + H_{ZZ}, \qquad (5)$$

where we have *continuous* control over J_{ij}. As mentioned above, when the Josephson coupling J_{ij} is nonzero, the corresponding matrix elements necessarily vanish: $\Delta_i = \Delta_j = 0$. In the idle state, Δ_i is non-zero and the qubit undergoes coherent tunneling. We proceed to show how universal QC can be performed given the outlined constraints.

III. ENCODING, INITIALIZATION, AND ENCODED LOGIC GATES

A. Code

In Ref. [32], it was shown how to perform universal quantum computation on the dGB qubit in the case where individual control of the bias b_i is possible. Here, we do not allow for this possibility and, since H_X and H_{ZZ} are insufficient for universal QC on the physical qubits, the techniques of Ref. [32] will have to be supplemented. This is achieved by encoding a pair of physical qubits as one logical qubit in the following way:

$$\begin{aligned}
|0_L\rangle_m &= |0\rangle_{2m-1} \otimes |1\rangle_{2m} \equiv |0_{2m-1}1_{2m}\rangle \\
|1_L\rangle_m &= |1\rangle_{2m-1} \otimes |1\rangle_{2m} \equiv |1_{2m-1}1_{2m}\rangle
\end{aligned} \qquad (6)$$

for the mth logical qubit with $m = 1, 2, ..., N/2$ and where there are a total of N physical qubits. It is easy to check that any other encoding into two qubits is not
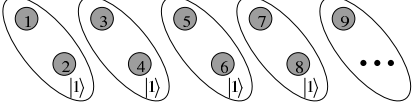
FIG. 2: Triangular arrangement of physical qubits, with the ancilla qubits in the bottom row. Logical qubits are formed by pairs of top row and ancilla qubits, as indicated by the ovals. The ancilla qubits should all be kept in the $|1\rangle$ state.

preserved under $H_X + H_{ZZ}$. Moreover, if σ^x_{2m} terms are allowed to act then the encoding of Eq. (6) is not preserved either. We address this issue below.

B. Initialization

Initialization of each encoded qubit can be done by measurement. This will project on the physical qubit's computational basis. Hence measuring the $(2m-1)^{\text{th}}$ physical qubit will yield either $|0\rangle_{2m-1}$ or $|1\rangle_{2m-1}$ (either one would do), while measuring the $(2m)^{\text{th}}$ physical qubit will have to be repeated until the outcome $|1\rangle_{2m}$ is obtained. Since tunneling will cause bit-flips immediately following the measurement, a good strategy is to simultaneously measure both these physical qubits, and repeat the measurement of the $(2m-1)^{\text{th}}$ physical qubit (projecting on the computational basis every time) until that of the $(2m)^{\text{th}}$ physical qubit has converged on $|1\rangle_{2m}$. Alternatively, initialization can be performed by field-cooling the qubits. By choosing the proper orientation for the external field, all physical qubits can be prepared in the $|1\rangle$ state and, therefore, all logical qubits in the $|1_L\rangle$ state. This strategy will most probably not be of practical use during computation but can serve to provide an initial supply of fresh qubits. For the dGB qubits, a current in terminal B can also be used to initialize all qubits to the $|1_L\rangle$ state [31].

C. Single-Qubit Gates

On the logical qubits, the encoded single-qubit operations are:

$$
\begin{aligned}
X_m &= \sigma^x_{2m-1} \\
Z_m &= \sigma^z_{2m-1} \otimes \sigma^z_{2m}.
\end{aligned}
$$

X_m acts as a logical σ^x operation on the encoded qubit, since:

$$
X_m|0_L\rangle_m = \sigma^x_{2m-1}|0_{2m-1}1_{2m}\rangle = |1_{2m-1}1_{2m}\rangle = |1_L\rangle_m.
$$

Similarly, it is simple to check that Z_m acts as an encoded phase shift. Note that whenever X_m is on Z_m is off, and *vice versa*, so that we can easily implement an Euler angle rotation [Eq. (9) below] to generate the Lie group $SU(2)$ on the mth logical qubit.

D. Two-Qubit Gates

It can be shown that the present model is not universal with nearest-neighbor interactions only in 1D. Therefore we consider a quasi-1D triangular arrangement, as shown in Figure 2. Each dot represents a physical qubit and the numbers are our indexing scheme: odd-numbered qubits are in the top row. In this arrangement qubits $2m-1$, $2m$, $2m+1$ are all nearest-neighbors and are connected by SSETs. The logical qubits are represented by pairs, as indicated by the ovals in Figure 2. The lower line of physical qubits are all in the state $|1\rangle$. We refer to these as the ancilla qubits.

As detailed in Ref. [32], interaction between physical qubits is supplied by the SSETs which couples pairs of qubits through a term $J_{ij}\sigma^z_i \otimes \sigma^z_j$. With the above encoding, an encoded controlled-phase gate (CPHASE) between the mth and $(m+1)^{\text{th}}$ logical qubits corresponds simply to coupling the odd-numbered (top-row) physical qubits of those two logical qubits. To see this, note that the Hamiltonian generating the CPHASE gate is

$$
\begin{aligned}
H^{\text{CPHASE}}_{m,m+1} &= J_{2m-1,2m+1}Z_m \otimes Z_{m+1} \\
&= J_{2m-1,2m+1}\sigma^z_{2m-1} \otimes \sigma^z_{2m+1}. \quad (7)
\end{aligned}
$$

Hence the implementation of the CPHASE gate only involves a (quasi 1D-)nearest-neighbor two-body term, and the control of the single Josephson energy $J_{2m-1,2m+1}$. Only odd-numbered physical qubits are involved in this gate.

We now have all the ingredients for universal QC. However, there are some subtleties and potential simplifications, to which we now turn.

IV. ALTERNATIVE CODE USING A BUS QUBIT

Since the even-numbered qubits in Figure 2 must be kept in an identical state there is the possibility of simply replacing them all by a *single* "bus" qubit which is kept in the $|1\rangle$ state. Thus instead of using N physical qubits we would use only $N/2 + 1$, as depicted in two possible geometries in Figure 3. It is simple to check that doing so is consistent with our logical operations: Since $X_m = \sigma^x_{2m-1}$ and $H^{\text{CPHASE}}_{m,m+1} = J_{2m-1,2m+1}\sigma^z_{2m-1} \otimes \sigma^z_{2m+1}$, the only operation that involves the even-numbered qubits is $Z_m = \sigma^z_{2m-1} \otimes \sigma^z_{2m}$. If we replace the $(2m)^{\text{th}}$ qubit by a fixed bus qubit $|1\rangle_b$ then the only change necessary is $Z_m = \sigma^z_{2m-1} \otimes \sigma^z_b$. I.e., we need to be able to turn on/off a Josephson coupling between all odd-numbered qubits and the bus qubit. We remark that the situation where a qubit in a code must be kept fixed has also arisen in the context of encoded universality for the XY model, namely the "truncated qubit" of Ref. [16].
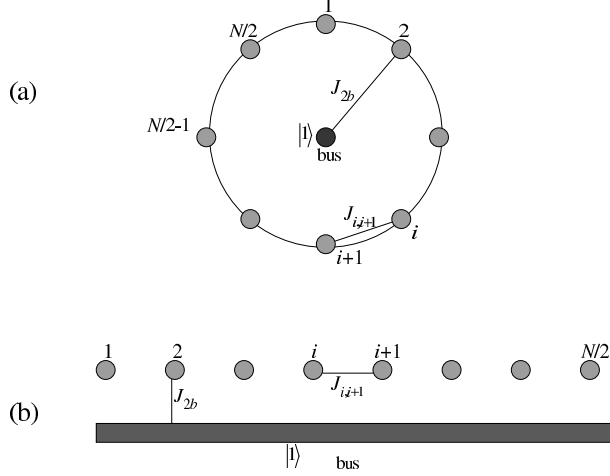
FIG. 3: Architecture with single bus qubit. Two possible arrangements are shown, with bus qubit at center of circularly arranged qubits (a), or "long" bus qubit connected to all other qubits (b). The bus qubit should always be in the $|1\rangle$ state.

V. PROBLEMS DUE TO TUNNELING ON PASSIVE QUBITS AND ENCODED SELECTIVE RECOUPLING

There is a problem with performing universal QC using only $H_X + H_{ZZ}$ and the above encoding, which must now be addressed. As stressed above, in the idle state each qubit undergoes Rabi oscillations. We therefore need to "freeze" the evolution of passive qubits while logical operations are performed on active qubits. In Ref. [32] this was done by using recoupling pulses, a technique similar to that used routinely in NMR [34, 35]. However, this was implemented by single-qubit bias pulses, which, as stressed above, we do not assume are available here. To solve this problem, we therefore extend the scheme of Ref. [32] to the technique of *encoded selective recoupling* developed in Ref. [13].

Like selective recoupling, encoded selective recoupling is based on some simple identities and the elementary theory of angular momentum. Let A and B be arbitrary Hermitian operators, θ, ϕ real numbers. Then

$$e^{-i\varphi A}e^{i\theta B}e^{i\varphi A} = e^{i\theta \exp(-i\varphi A)B\exp(i\varphi A)}. \qquad (8)$$

Now assume $A = \hat{n} \cdot \vec{J}$ where $\hat{n}$ is a real 3D unit vector and $\vec{J} = (J_x, J_y, J_z)$ is an angular momentum vector operator. Then $R \equiv \exp(-i\varphi A)$ is a rotation operator that can be written as a product of three Euler angle rotations [36, Sec. 13]:

$$R = e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}, \qquad (9)$$

where α, β, γ are the Euler angles. If we take B to be an irreducible tensor operator of rank L (i.e., a member of a set of $2L+1$ functions T_{LM}, $M = -L, -L+1, ..., L$, which transform under the $2L+1$-dimensional representation of

the rotation group), then [36, Sec. 17]:

$$RT_{LM}R^{\dagger} = \sum_{M'} D^L_{M'M}(\alpha, \beta, \gamma)T_{LM'},$$

where D are the Wigner rotation matrices, whose matrix elements $D^L_{M'M}(\alpha, \beta, \gamma)$ are the matrix elements of R in the LM representation. The Wigner matrices are extensively tabulated [36], so that in principle calculating the transformation $\exp(-i\varphi A)B\exp(i\varphi A)$ appearing in Eq. (8) is always possible for angular momentum operators A, B. Let us consider a few cases of importance to us here, and of general interest in quantum computing.

Assume that A and B are anticommuting operators, e.g., different Pauli matrices. Let I be the identity operator. Then, we define the operation of "conjugation by A", i.e., Eq. (8), as [37]:

$$
\begin{aligned}
C_A^{\varphi} \circ e^{i\theta B} &\equiv e^{-i\varphi A}e^{i\theta B}e^{i\varphi A} \\
&= \exp[i\theta(B\cos(2\varphi) + iBA\sin(2\varphi))] \\
&= \begin{cases} e^{-i\theta B} & \text{if } \varphi = \pm\pi/2 \\ e^{\pm i\theta(iBA)} & \text{if } \varphi = \pm\pi/4 \\ e^{i\theta B(I \pm iA)/\sqrt{2}} & \text{if } \varphi = \pm\pi/8 \end{cases} . \quad (10)
\end{aligned}
$$

To prove these relations one can use the Wigner D matrices as above, or, more simply but less generally, note that for anticommuting A and B that in addition satisfy $A^2 = I$:

$$
\begin{aligned}
e^{-i\varphi A}Be^{i\varphi A} &= (I\cos\varphi - Ai\sin\varphi)B(I\cos\varphi + Ai\sin\varphi) \\
&= B\cos^2\varphi + ABA\sin^2\varphi - i\sin\varphi\cos\varphi[A, B] \\
&= B\cos(2\varphi) + iBA\sin(2\varphi).
\end{aligned}
$$

The result of Eq. (10) with $\varphi = \pi/2$ is used in the NMR technique of refocusing, or more generally, selective recoupling [34, 35]. It allows one to flip the sign of a term in a Hamiltonian, which can be used to cancel unwanted evolution. The result with $\varphi = \pi/4$ can be viewed as a special case of Euler angle rotations which preserve a discrete group (in QC commonly the Pauli group – the group of tensor products of Pauli matrices). The case with $\varphi = \pi/8$ allows us to move from the Pauli group to the group algebra of the Pauli group, and is useful for rotating sums of operators into a desired direction on the Bloch sphere (see Section VIII B below). The conjugation method can be used on the physical as well as the encoded qubits, in which case we refer to it as "*encoded selective recoupling*".

Note that in order to implement e^{-itA} where A is a Hamiltonian that is turned on for a time t, we need to find ϑ such that $e^{i\vartheta A} = I$ and implement $e^{i(\vartheta-t)A}$ instead. This depends on A having rationally related eigenvalues, which is the case for the Hamiltonians of interest to us. E.g., for $A = J\sigma_1^z \otimes \sigma_2^z$ we have $\vartheta = 2\pi/J$, so that $\exp(-it(-J)\sigma_1^z \otimes \sigma_2^z) = \exp(i(t-2\pi/J)J\sigma_1^z \otimes \sigma_2^z)$. I.e., if this Hamiltonian is applied for a time $t - 2\pi/J > 0$ it effectively evolves as if $J \to -J$. This method circumvents

the need for switching the sign of the Hamiltonian itself, which is difficult to realize in the case of the Josephson coupling J_{ij}.

We are now ready to address and solve the issue of the idle qubit Rabi oscillations mentioned above. We will first concentrate on the realization of Figure 2 before turning to the bus qubit implementation.

A. Leakage

In the idle state there will be tunneling on all qubits implementing unwanted σ^x operations. On the even-numbered qubits of Figure 2, these bit flips cause a transition out of the computational space, or *leakage*.

Our goal is then to eliminate the leakage term on all even-numbered qubits

$$\Delta = \sum_{m=1}^{N/2} \Delta_{2m} \sigma^x_{2m}$$

due to the free evolution of these qubits during an idle period. For simplicity consider just the first and second encoded qubits, i.e., the term $\Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4$. We can refocus it using $\sigma^z_2 \otimes \sigma^z_4$, which will flip the sign of the offending term. To do so we need to turn on $\sigma^z_2 \otimes \sigma^z_4$ for a time τ_{24} such that $\tau_{24} J_{24} = \pm\pi/2$ [recall Eq. (10)]. Specifically, since $A = \sigma^z_2 \otimes \sigma^z_4$ and $B = \Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4$ are Hermitian and anticommute:

$$e^{it(\Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4)} \left(C^{\pi/2}_{\sigma^z_2 \otimes \sigma^z_4} \circ e^{it(\Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4)} \right) = I,$$

so that evolution under $\Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4$ has been eliminated.

Note that Eq. (10) implicitly assumes that while A is on B is off, and *vice versa*. This is satisfied in our procedure, since the tunneling term $\Delta_2 \sigma^x_2 + \Delta_4 \sigma^x_4$ is on (off) while the coupling term $\sigma^z_2 \otimes \sigma^z_4$ is off (on). Interestingly, this is different from most other recoupling schemes, e.g., [13, 26], where one typically assumes strong pulses that completely dominate the natural evolution (see [38] for an analysis of this issue). Continuing, Δ can be completely eliminated if all pairs of qubits $\{(2,4),(6,8),...\}$ are refocused in a similar manner, i.e., by turning on $\sigma^z_{4m-2} \otimes \sigma^z_{4m}$ for a time $\tau_{4m-2,4m}$ such that $\tau_{4m-2,4m} J_{4m-2,4m} = \pm\pi/2$ (where $m = 1, 2, ..., N/4$). With J_{ij} of the order of the GHz, the refocusing pulses will typically be of the order of the nanosecond. These operations need to be applied in parallel, which is possible because they commute (they are applied to different qubits). This means that all $J_{4m-2,4m}$ must be rationally related, so that appropriate intervals $\tau_{4m-2,4m}$ can be found. Obviously, a possibility is to set all $J_{4m-2,4m}$ equal but this could be difficult to realize in practice.

B. Unwanted σ^x Operations During Encoded Single Qubit Operation

We now take care not only of leakage out of the code subspace, but errors due to tunneling on passive qubits during operation of encoded single qubit gates on active qubits.

For example, while $X_l = \sigma^x_{2l-1}$ is on we still have tunneling on all qubits other than $2l-1$, again implementing unwanted σ^x operations. To emphasize this let us rewrite H_X as:

$$\begin{aligned} H_X &= H_0 + H_{\bar{X}} + \Delta \\ &\equiv \Delta_{2l-1} X_l + \sum_{m \neq l}^{N/2} \Delta_{2m-1} X_m + \sum_{m=1}^{N/2} \Delta_{2m} \sigma^x_{2m} \end{aligned}$$

where H_0 is the desired evolution, $H_{\bar{X}}$ are unwanted bit flips on the passive qubits and Δ is a leakage term.

Similarly, while Z_l is on, we have tunneling in all qubits other than $2l-1$ and $2l$, again implementing unwanted bit flips. I.e., the Hamiltonian while Z_l is on is:

$$H_{Z_l} = J_l Z_l + \sum_{m \neq l}^{N/2} \Delta_{2m-1} X_m + \sum_{m \neq l}^{N/2} \Delta_{2m} \sigma^x_{2m} \qquad (11)$$

Thus we either have to implement computational operations on the other encoded qubits ($m \neq l$), or eliminate this unwanted evolution.

We start by first taking care of the single encoded $X_l = \sigma^x_{2l-1}$ operation in the presence of tunneling on all other qubits. I.e., we need to eliminate not just the leakage term Δ but also the unwanted logical operations $H_{\bar{X}}$. Suppose we wish to implement X_1 for a time T, i.e., the operation $e^{-iT\Delta_1 X_1}$. The solution is shown schematically in Figure 4. We refocus all other qubits in pairs $\{(m,n)\} = \{(2,3),(4,5),(6,7),(8,9),...\}$, using the same idea as for the leakage problem. Namely, we turn on the interactions $\{\sigma^z_m \otimes \sigma^z_n\}$ for times τ_{mn}, τ''_{mn} such that $\tau_{mn} J_{mn} = \pi/2$ and $\tau''_{mn} J_{mn} = 3\pi/2$ (we are now explicitly taking into account the fact that we cannot switch the sign of J_{mn}). In between and after the $\pi/2$ and $3\pi/2$ periods these pairs of qubits are allowed to evolve freely for times τ'_{mn} under the tunneling terms $\{\Delta_m, \Delta_n\}$. The condition that must then be satisfied is:

$$\tau_{mn} + 2\tau'_{mn} + \tau''_{mn} = T.$$

Since, e.g., $\tau_{23} J_{23} = \pi/2$, the free evolution time is determined by $\tau'_{23} = (T - 4\tau_{23})/2 = T/2 - \pi/J_{23}$, or, in general [57]:

$$\tau'_{mn} = T/2 - \pi/J_{mn}.$$

These times must be positive, so that we must be able to make J_{mn} (assuming it is positive) large enough that
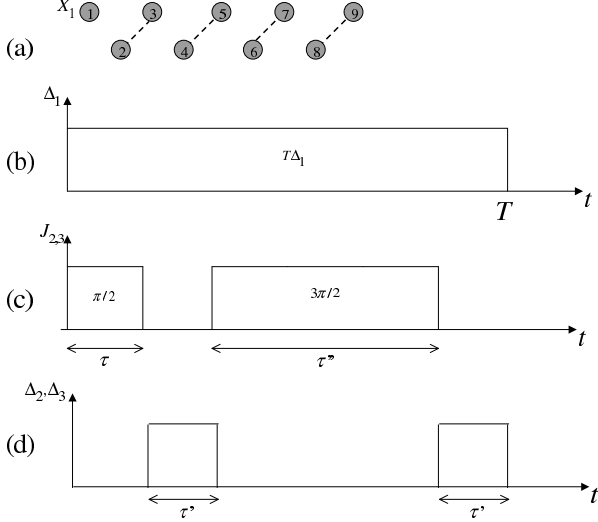
$$J_{mn} > 2\pi/T.$$

FIG. 4: Single logical-qubit X operation. Shown in (a) is the algorithm for the implementation of X_1, with recoupling connections indicated by dashed lines. (b) Shows a rectangular pulse with an area $T\Delta_1$ implementing the logical operation $\exp(-iT\Delta_1 X_1)$. (c),(d) Show the corresponding recoupling sequence on qubits $2,3$ (exactly the same diagrams apply to the other qubit pairs $4,5$ etc.). E.g., in (c) the first pulse lasts for a time τ such that $\tau J_{2,3} = \pi/2$, immediately followed by a tunneling period τ' (assuming $\Delta_2 = \Delta_3$), etc.

Conversely, if J_{mn} has a maximum value $J_{mn}^{\max}$ then there is a time

$$T_{\min} = 2\pi/J_{mn}^{\max}$$

below which we cannot apply an encoded logical X operation. Thus all encoded logical X operations must be implemented as $e^{-iT\Delta_1 X_1}$ with $T \geq T_{\min}$. This means that the smallest angle of rotation around the encoded x axis is $\theta_{\min} = 2\pi\Delta_{m(n)}/J_{mn}^{\max}$. With $\Delta_m \sim 100\text{MHz}$ [31] and J_{mn} of the order of the GHz, $\theta_{\min}$ is of the order of $2\pi/10$ and can be made smaller if the Josephson energy J_{mn} is made larger. However, this is not necessary in the fault-tolerant "standard paradigm" of universal QC, where all logic gates can be built up from the Hadamard, $\pi/8$ and CPHASE gates, which require θ of $\pi/2$, $\pi/8$, and $\pi/4$ respectively [3]. Of course, here we have in mind an *encoded* version of this universal set of gates.

The solution to the single encoded $Z_l = \sigma_{2l-1}^z \otimes \sigma_{2l}^z$ operation in the presence of tunneling on all other qubits, Eq. (11), is almost identical to that of single encoded X operations. Suppose we wish to implement $Z_1 = \sigma_1^z \otimes \sigma_2^z$ for a time T, i.e., the operation $e^{-iT J_1 Z_1}$. The solution is shown schematically in Figure 5. We refocus all other qubits in pairs $\{(m,n)\} = \{(3,5),(4,6),(7,9),(8,10),...\}$ by again turning on the interactions $\{\sigma_m^z \otimes \sigma_n^z\}$ for times τ_{mn}, τ_{mn}'' such that $\tau_{mn} J_{mn} = \pi/2$ and $\tau_{mn}'' J_{mn} = 3\pi/2$. All other details are the same as above.
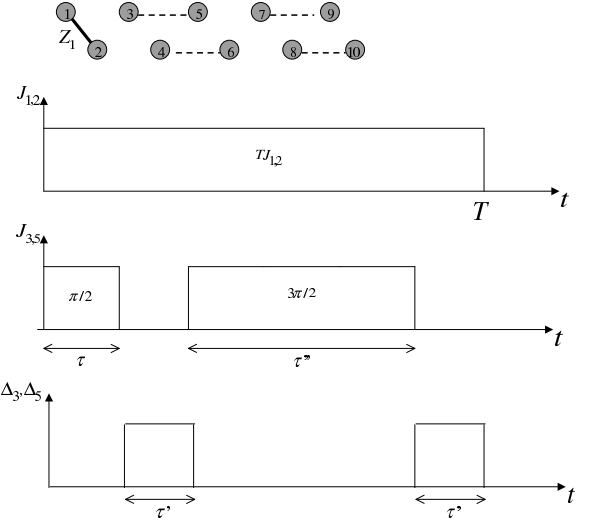
FIG. 5: Single logical-qubit Z operation. Other details as in Figure 4.

C. Unwanted σ^x Operations During Two Qubit Encoded $Z \otimes Z$ Operation

The problem is the same as in the previous cases. We now have to deal with the Hamiltonian

$$H_{Z_l Z_{l+1}} = J_{2l-1,2l+1} Z_l Z_{l+1} + \sum_{m \neq l,l+1}^{N/2} \Delta_{2m-1} X_m + \Delta,$$

$$(12)$$

i.e., leakage plus unwanted X operations on all but encoded qubits $l, l+1$. The solution is completely analogous to the case of single encoded Z operations.

D. Compatibility of Recoupling Sequences

There is still one problem with the above method that needs to be solved: recoupling in the single encoded X case requires pairing up $N-1$ physical qubits, whereas recoupling in the single encoded Z and encoded $Z \otimes Z$ cases requires pairing up $N-2$ physical qubits. Thus in the former case we would want N to be odd, while in the latter we need N to be even. Assuming N to be even, there are several possible solutions to this.

One solution is to reintroduce single-qubit σ^z operations, but only on (say) the last qubit. This qubit can then be refocused independently. This is however not very elegant (or practical) as our aim was to eliminate single-qubit σ^z operations in the first place.

A second possibility is to make the Josephson coupling energy much larger than the tunneling matrix element. In this case, it is possible to neglect the evolution of the passive qubits due to tunneling during the on-time of the Josephson coupling on the active qubit. Reaching
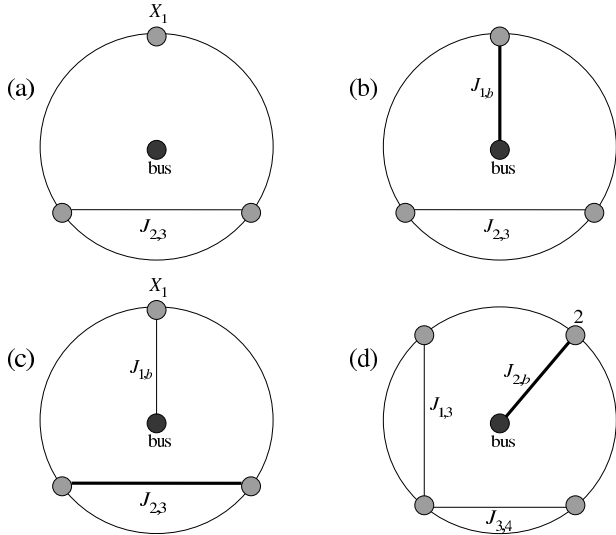
FIG. 6: Logic operations with recoupling connections in bus qubit architecture. Recoupling works well with an odd number of non-bus qubits, as shown in (a)-(c). (a) Single logical-qubit X_1 operation, qubits $2, 3$ recoupled. Here it is essential that the bus qubit can be kept fixed in the $|1\rangle$ state by a method other than recoupling; (b) Single logical-qubit Z_1 operation, qubits $2, 3$ recoupled; (c) Two logical-qubit CPHASE operation, qubit 1 recouples with bus qubit. With an even number of non-bus qubits there is a problem, as shown in (d): it is not possible to implement a single logical-qubit Z_2 operation because three non-bus qubits have to be recoupled.
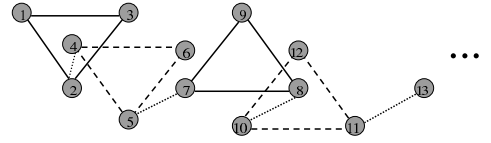

FIG. 7: 3D architecture using 3 physical qubits to encode 2 logical qubits. Two triangular 2D arrays are superimposed (alternating solid and dashed lines), such that qubits 1,2,3,4 form a tetrahedron, likewise for qubits 3,6,7,9, and 8,10,11,12, etc. Pairs of logical qubits correspond to physical qubits 123, 456, 789, etc. This allows qubits 12,23,13,24 to have nearest neighbor interactions, so that the logical operations Z_1, Z_2, $Z_1 Z_2$, $Z_2 Z_3$ can all be implemented through nearest neighbor couplings. Likewise, qubits 45,56,46,57 are nearest neighbors, allowing logical operations Z_3, Z_4, $Z_3 Z_4$, $Z_4 Z_5$ to be implemented through nearest neighbor couplings. The required connections between qubits belonging to different planes are shown by short-dashed lines.

the necessary constraint $|J_{ij}| \gg |\Delta_k| \, \forall k$ can however be difficult in practice and, even if it is reached, there will be accumulation of small errors on the passive qubits. This solution is therefore not the most practical either.

The simplest way to solve the compatibility problem is to assume that we can fix (say) the last qubit (number N) in the $|1\rangle$ state. This corresponds to producing that qubit with a very high tunneling barrier and initializing it in the $|1\rangle$ state. In this manner this qubit is "frozen" and the problem of unwanted evolution on it does not arise. For the dGB qubit, producing a qubit with a high tunneling barrier can be done by choosing proper misalignment between the two d-wave superconductors or, more simply, by working with larger junctions [31].

An interesting alternative is to extend this concept of "frozen" qubit to all ancilla qubits. In this case, there is no possible leakage out of the code subspace. Moreover, this applies well to the concept of "bus qubit" presented above. In this case, the bus is a single qubit with a large tunneling barrier and initialized to the $|1\rangle$ state.

Considering the bus qubit case, even if there is no leakage, the problem of freezing passive qubits is still present. Fortunately, the solution explained above still applies. However, as illustrated in Figure 6, with a bus qubit the number of physical qubits should be odd. The reason is that otherwise, during the implementation of, e.g., $Z_1 = \sigma_1^z \otimes \sigma_b^z$ we would be left with an odd number

of qubits to be refocused [Figure 6(d)]. As seen in Figure 6((a)-(c)), with an odd number of logical qubits all logic operations can be performed provided the bus qubit can be kept fixed in the $|1\rangle$ state during single qubit X operations [Figure 6(a)].

The question of which of the two designs, bus qubit or with $N/2$ ancilla qubits, is superior, will be decided by engineering constraints. The bus qubit imposes a circular geometry [as in Figure 3(a)], or has to be made long [as in Figure 3(b)], or some other method has to be found to connect it to all other qubits. In particular, it can be challenging experimentally to connect many qubits to the bus. Having several bus qubits and, for example, repeating the circular geometry of Figure 3(a) in an triangular lattice could help in reducing these constraints.

VI.   CODES WITH HIGHER RATES

So far we have considered encoding a single logical qubit into two physical qubits. This code has a rate of $1/2$. In this section we consider codes with higher rates. Specifically, we propose an encoding of two logical qubits into three physical qubits, yielding a rate of $2/3$.

Consider the following encoding:

$$
\begin{aligned}
|0_L\rangle \otimes |0_L\rangle &= |0_1 0_2 0_3\rangle \\
|0_L 1_L\rangle &= |010\rangle \\
|1_L 0_L\rangle &= |100\rangle \\
|1_L 1_L\rangle &= |110\rangle.
\end{aligned}
$$

The third physical qubit is always 0 here, so once again the bus qubit idea applies. The corresponding logical

operations are:

$$X_1 = \sigma_1^x \quad X_2 = \sigma_2^x$$
$$Z_1 = \sigma_1^z \sigma_3^z \quad Z_2 = \sigma_2^z \sigma_3^z$$
$$Z_1 Z_2 = \sigma_1^z \sigma_2^z,$$

as is readily checked. E.g.,

$$Z_1 Z_2 |0_L 0_L\rangle = \sigma_1^z \sigma_2^z |000\rangle = |000\rangle = |0_L 0_L\rangle$$
$$Z_1 Z_2 |0_L 1_L\rangle = \sigma_1^z \sigma_2^z |010\rangle = -|010\rangle = -|0_L 1_L\rangle,$$

etc. Note that Z_1 and Z_2 explicitly use the third qubit, showing that it is essential. As before, we think of these logical operations as Hamiltonians, not gates. We therefore have a universal generating set for two logical qubits encoded into the first three physical qubits. To complete the universal set we also need to be able to couple logical qubits belonging to different blocks of three physical qubits. Let physical qubits $4, 5, 6$ encode the next pair of logical qubits. Then

$$Z_2 Z_3 = \sigma_2^z \sigma_4^z$$

is a logical operation coupling the two blocks. To verify this consider the truth table for this operation. Let x, y be 0 or 1. Then:

$$Z_2 Z_3 |x_L 0_L\rangle |0_L y_L\rangle = \sigma_2^z \sigma_4^z |x_1 0_2 0_3\rangle |0_4 y_5 0_6\rangle = |x_1 0_2 0_3\rangle |0_4 y_5 0_6\rangle = |x_L 0_L\rangle |0_L y_L\rangle$$
$$Z_2 Z_3 |x_L 0_L\rangle |1_L y_L\rangle = \sigma_2^z \sigma_4^z |x_1 0_2 0_3\rangle |1_4 y_5 0_6\rangle = -|x_1 0_2 0_3\rangle |1_4 y_5 0_6\rangle = -|x_L 0_L\rangle |1_L y_L\rangle$$
$$Z_2 Z_3 |x_L 1_L\rangle |0_L y_L\rangle = \sigma_2^z \sigma_4^z |x_1 1_2 0_3\rangle |0_4 y_5 0_6\rangle = -|x_1 1_2 0_3\rangle |0_4 y_5 0_6\rangle = -|x_L 1_L\rangle |0_L y_L\rangle$$
$$Z_2 Z_3 |x_L 1_L\rangle |1_L y_L\rangle = \sigma_2^z \sigma_4^z |x_1 1_2 0_3\rangle |1_4 y_5 0_6\rangle = |x_1 1_2 0_3\rangle |1_4 y_5 0_6\rangle = |x_L 1_L\rangle |1_L y_L\rangle,$$

which is the desired action. Note that next-nearest neighbor couplings are involved in Z_1 (inside a block of three) and $Z_2 Z_3$ (between blocks of three). Figure 7 shows a possible arrangement in 3D. Two triangular arrays of qubits are superimposed such that the relevant qubits form tetrahedrons (see caption). With this particular geometry, only nearest neighbor couplings are required. Qubit arrays of lower dimensionality will require next-nearest neighbor couplings, which may or may not be easier to implement experimentally. For definiteness we continue the discussion here while referring to the 3D layout, but it should be kept in mind that a 1D/2D layout with overlapping wires may prove to be advantageous.

Recoupling will still be needed since we still have passive tunneling on the idle physical qubits. The recoupling paths are shown in Figure 8. As in the rate 1/2 code, there is an incompatibility between the number of qubits required for recoupling in the case of a logical X operation, and the number required for the logical Z and ZZ operations. This can be solved by introducing an auxiliary qubit. This qubit then has to be frozen in the $|0\rangle$ state, as discussed above.

All the considerations above can be modified easily to the case of a single bus qubit, taken to be, e.g., a long single qubit fixed in the $|0\rangle$ state (i.e., the single bus qubit now replaces all physical qubits numbered $3n$, $n = 1, 2, ...$ in the discussion above). This architecture is shown in Figure 9. The recoupling paths are shown in Figure 10 (compare to the recoupling figures for the one-in-two encoding). However, note that with a "global" bus qubit the advantage of a higher rate disappears. As in the above encoding of one logical qubit per two physical qubits, with a global bus qubit optimal use of all physical
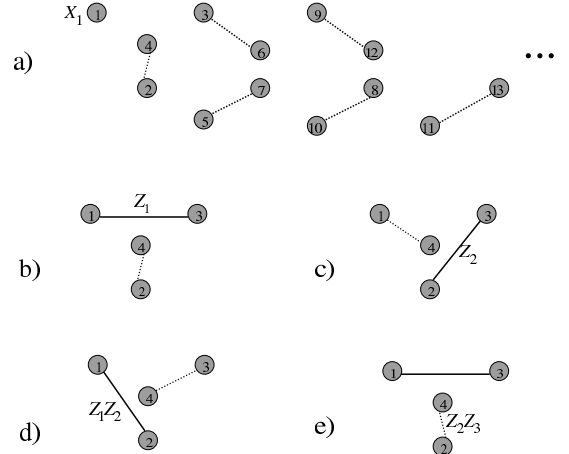

FIG. 8: (a) Implementation of logical X_1 operation on logical qubit 1 is done by allowing physical qubit 1 to undergo free tunneling. This requires recoupling all other physical qubits in pairs as indicated by short-dashed lines. (b) Implementation of logical Z_1 operation on logical qubit 1 is done by Josephson coupling physical qubit 1 to its ancilla qubit, number 3. This requires recoupling all other physical qubits in pairs as indicated by dashed lines. (c),(d),(e) Same as (b), for implementing logical operations Z_2, $Z_1 \otimes Z_2$, $Z_2 \otimes Z_3$.

qubits has already been made.

There will be codes with even higher rates (e.g., use four physical qubits to encode three logical qubits by fixing the fourth physical qubit). However, in this case the geometrical constraints may become impossible to satisfy even in 3D if we can only use nearest-neighbor coupling.
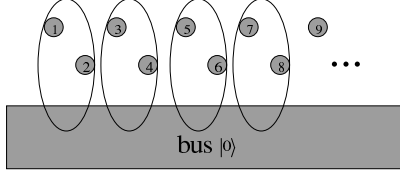
FIG. 9: Architecture using 3 physical qubits to encode 2 logical qubits, using a global bus qubit. Ovals indicate grouping, i.e., blocks of three physical qubits are formed by physical qubit 1,2,bus; 3,4,bus; etc. The bus is fixed in the $|0\rangle$ state.
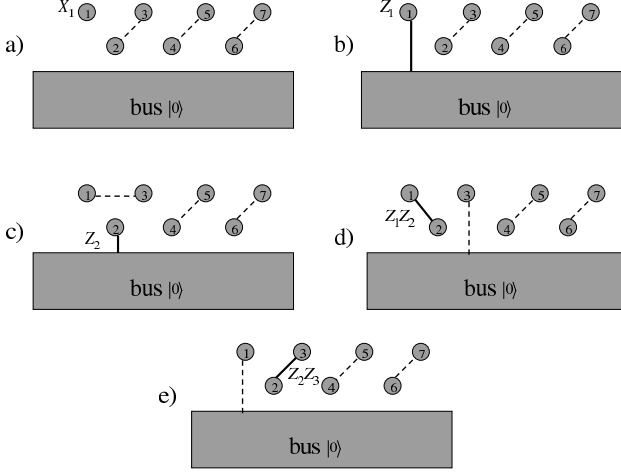

FIG. 10: (a) Implementation of logical X operation on logical qubit 1 is done by allowing physical qubit 1 to undergo free tunneling. This requires recoupling all other physical qubits in pairs as indicated by dashed lines. (b) Implementation of logical Z operation on logical qubit 1 is done by Josephson coupling physical qubit 1 to bus. This requires recoupling all other physical qubits in pairs as indicated by dashed lines. Similarly for (c),(d), and (e).

Therefore such higher rate codes will require introducing longer-range coupling. What the tradeoffs are between a 3D arrangement and long-range couplings, is, again, an implementation-dependent question that we cannot answer here.

VII. IMPLEMENTATION AND CONNECTION TO OTHER DESIGNS

We now turn to a discussion of the usefulness and practicality of the above scheme to the dGB qubits and to the other superconducting phase-qubits in general.

For the dGB qubits, the usefulness of the above scheme is clear. The application of individual qubit bias following the scheme of Refs. [31, 32] would be rather difficult experimentally. The encoding and bus qubit proposed here circumvent this problem. Moreover, for the dGB qubits, the implementation of the bus qubit concept turns
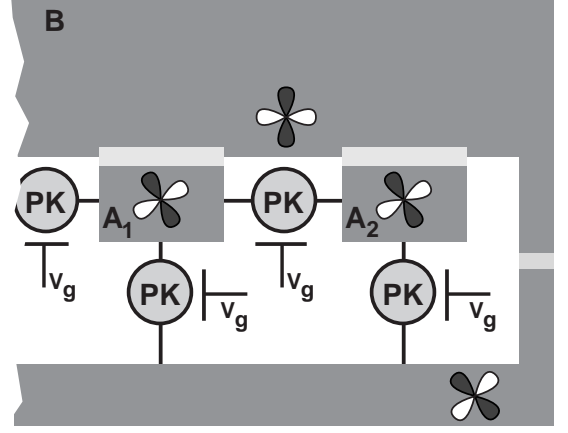

FIG. 11: Superconducting grain boundary qubits with "bus qubit". The qubits are connected to the bus by SSETs.

out to be a rather simple modification of the layout already presented in [31, 32]. As shown in Figure 11, the bus can be a large piece of d-wave superconductor coupled to terminal B of the qubits by a weak link. If this bus is large and has the proper misalignment of its order parameter with respect to terminal B, it will have a fixed phase and hence correspond to a fixed logical state. This bus is coupled to the other qubits by SSETs to provide the necessary J_{ij} couplings. Thus, based on the encoding of Eq. (6) which suggested the use of a bus qubit, the equivalent of single-qubit bias operations could be reintroduced in a way that is experimentally simpler than what was previously suggested [31, 32].

The concepts explored in this paper can be applied to the other superconducting phase qubit designs [33, 39, 40, 41]. First, for the rf-SQUID like design of Ref. [39, 40], single-qubit logical operations are provided by control of external fluxes in two different loops of the SQUID, Figure 12a). One of these fluxes controls the asymmetry of the qubit's potential energy landscape (i.e., σ_i^z operations), while the other controls the tunnel barrier (i.e., σ_i^x operations). Qubits can be coupled inductively to provide the necessary two-qubit logical gates. These coupling can be arranged to provide a term in the Hamiltonian which has the same symmetry as considered above, $J_{ij}\,\sigma_i^z \otimes \sigma_j^z$ [40].

Consider moving to a design with only 3 junctions and one loop (as was used experimentally in [28]), such that only the asymmetry can be controlled, Figure 12b). By choosing the Josephson energies such that in the idle state coherent tunneling is possible, the Hamiltonian describing the system is (5), with the same constraints as for the dGB qubit. The encoding and results presented here therefore apply equally to these qubits and provide enough control to perform universal QC. Of course, the application of the logical operations is more laborious than in the original 4-junction design but this has other advantages. First, there is some simplification in fabrica-
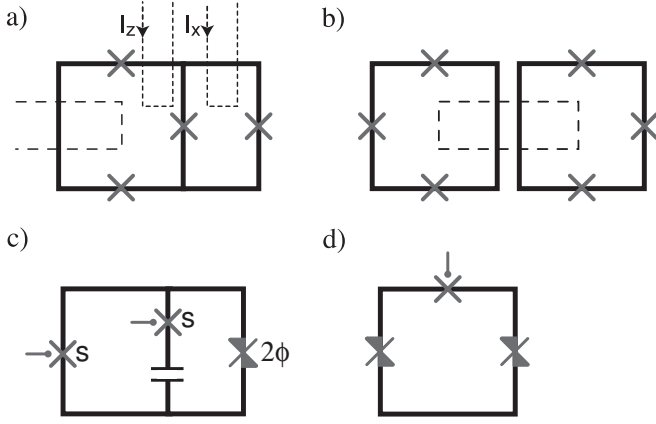
FIG. 12: (a) Original persistent-current qubit design. Currents I_z and I_x provide flux in large and small loop respectively and are used to control single-qubit operations. Coupling to a second qubit (not shown) can be inductive (long-dashed line). (b) Using the encoding of Eq. (6), a pair of coupled physical qubits can be used as one logical qubit. Unnecessary control lines have been discarded, keeping only the inductive two-qubit coupling. (c) 'Quiet qubit' design. The 2ϕ-junction realizes the two-level system. Single qubit operations are implemented by voltage pulses on the switches 's'. Coupling to other qubits is not shown. (d) As in (b), only coupling between a pair of physical qubits is retained to obtain one logical qubit.

tion. Moreover, the external flux controlling the tunnel barrier is a source of noise to the system. Removing such a source of decoherence in a system which has a small coherence time to begin with, is clearly advantageous. For this design, rather than the simplification in fabrication, the possibility of reducing sources of noise is the real advantage of using the encoding (6).

Similarly, for the "quiet qubit" design of Ref. [33], one could eliminate the external capacitor which is used to tune the tunneling amplitude, and choose the Josephson energies such that tunneling is possible in the idle state, Figure 12c). Connection to the strong $\pi/2-$junctions, used to control the asymmetry, can also be discarded, retaining only two-qubit couplings, Figure 12d). In this way the system again is described by (5) and the same ideas apply. As with the rf-SQUID design, use of the encoding (6) allows to remove some of the components of the quantum computer design while maintaining universality. We again stress that this reduces engineering constraints and removes some important sources of noise.

Implementation of the bus qubit for the "quiet qubit" design is similar to the phase bus discussed in Ref. [33]. Similar ideas can be used for the rf-SQUID like qubits.

By using the encoding suggested here, we are now relying mostly on two-qubit gates to perform QC. Therefore, qubit-qubit coupling should be easily controlled and the ratio of the characteristic energy when the qubit-qubit "switch" is on compared to when it is off should be high. These requirements are not different from the standard situation when the encoding (6) is not used. Further,

for this scheme to be useful in practice, the two-qubit gates should not be much slower than the 1-qubit gates. By choosing strong enough Josephson junctions for the SSETs, this should be satisfied for the dGB qubits. Similarly, in the design of Refs. [39, 40] the inductive coupling qubits should be of the order of single-qubit energies. The situation is similar for the "quiet qubit" design. All of the above remarks apply equally for the multi-terminal superconducting phase qubit [41].

VIII. REDUCTION OF ERRORS DUE TO DECOHERENCE

Standard quantum error correction techniques [42] are certainly compatible with the dGB qubit design. Here we focus on the recoupling [13, 34, 35, 43, 44] and the decoupling, or "bang-bang" (BB) method introduced in [45], and developed further, e.g, in Refs. [23, 24, 25, 46, 47, 48, 49, 50, 51, 52, 53]. The advantage of the recoupling and BB methods, is that they does not require extra qubit (space) resources, unlike quantum error correction [42] and/or decoherence-free subspaces [54, 55]. Specifically, we will show that decoherence can be reduced significantly using only the *existing controllable interactions*, acting on the encoded qubits. Different methods will be presented depending on the symmetry of the system-bath interaction. "Encoded decoupling" (i.e., acting on encoded qubits) of the type we discuss below has been previously suggested for solid-state [23, 24, 25] and NMR [26, 27] QC proposals. We further note that use of BB pulses for dGB qubits was previously discussed in Ref. [32], but access to single-qubit Z operations was assumed. We extend and generalize this discussion here.

The total Hamiltonian of a qubit system (S) and bath (B) can be written as

$$H = H_S + H_B + H_{SB},$$

where H_{SB} is the system-bath coupling. In order to use the BB method one makes two assumptions: First, (as in the threshold result of fault tolerant QC [4, 5]) that a controllable part of H_S can be made so strong that we can make H approximate that part of H_S to arbitrary precision. This is needed so that H_{SB} can be neglected during the pulse. Second, that one can pulse this controllable part of H_S with a pulse repetition time that is much shorter than the inverse of the bath frequency cut-off. This requirement arises since in between BB pulses the bath should not change, or else it will acquire phases that, when the bath is traced over, will result in decoherence in the system [45].

We will take the interaction term to have the linear form $H_{SB} = \sum_i \vec{\sigma}_i \cdot \vec{B}_i$, where $\vec{\sigma}_i$ is the vector of Pauli matrices acting on the system, and $\vec{B}_i$ are corresponding bath operators. This can represent the interaction of a qubit with a fluctuating control field. For example, for the superconducting phase qubit, a term $\sigma_i^z B_i^z$ arises due to fluctuation of local magnetic field. The relation

of H_{SB} to the parameters of the system and bath can be analyzed in detail [30]. Our approach to decoherence suppression depends on the time-scales that emerge from this analysis, and on the symmetry of the system-bath interaction.

A. Suppression of Axially Symmetric System-Bath Interaction by $Z \otimes Z$ Recoupling Method

Suppose that $|J_{ij}| \gg |\Delta_i|, |B_i^x|, |B_i^y|, |B_i^z|$ so that the strong parts of H_S are

$$H_{ij} = J_{ij}\sigma_i^z \otimes \sigma_j^z,$$

which we can turn on and off freely. If H_{SB} is of the general form $\sum_i \vec{\sigma}_i \cdot \vec{B}_i$, then H_{ij} obviously is not enough to eliminate H_{SB} by decoupling methods, since it commutes with the $\sigma_i^z B_i^z$ terms. However, by the same token, if the system-bath interaction has an *axial symmetry* so that

$$H_{SB} = \sum_i \left(\sigma_i^x B_i^x + \sigma_i^y B_i^y \right),$$

it can be eliminated by

$$\exp(-iH_{SB}t/2)\left[C_{\sum_{m=1}^{N/2} Z_m}^{\pi/2} \circ \exp(-iH_{SB}t/2) \right] = I,$$
(13)

where as before $Z_m = \sigma_{2m-1}^z \otimes \sigma_{2m}^z$ is the logical Z operation. Implicit in this calculation is that H_{SB} is negligible while Z_m is on. To the extent that this assumption breaks down there will be an error proportional to the ratio of the largest eigenvalue of H_{SB} by the smallest eigenvalue of H_S. The important point is that this type of axially symmetric system-bath interaction can be suppressed *without any extra resources*, simply by using the already available Josephson coupling. Moreover, the decoupling pulse $C_{\sum_{m=1}^{N/2} Z_m}^{\pi/2}$ used in Eq. (13) commutes with all $Z_{m'}$, $Z_{m'}Z_{m''}$ and with all $X_{m'}$ such that $m', m'' \neq m$. All these logical operations can therefore be executed in parallel with this decoupling procedure. However, since the decoupling pulses anticommute with X_m this logical operation is eliminated if it is turned on during decoupling. Hence we must alternate suppressing decoherence on the mth qubit and performing logical X operations on it. This implies that this qubit will suffer some decoherence while X_m is applied to it, unless we protect it by other means, such as active quantum error correction [42].

B. Suppression of Partial Symmetric System-Bath Interaction Hamiltonians by Decoupling

Suppose that $|J_{ij}| \gg |\Delta_i| \gg |B_i^x|, |B_i^y|, |B_i^z|$, so that we can freely turn $\Delta_i \sigma_i^x$ and $J_{ij}\sigma_i^z \otimes \sigma_j^z$ on and off, and while we do so H_{SB} becomes negligible. The only effect

of H_{SB} is to decohere the qubit system when it evolves freely, i.e., under $\exp(-it(H_B + H_{SB}))$.

Now suppose the system-bath interaction has a symmetry so that

$$H_{SB}^{yz} = \sum_i \left(\sigma_i^y + \sigma_i^z \right) B_i^{yz} + \sigma_i^x B_i^x.$$

Note that, using Eq. (10), a term of the form $\exp i\theta(\sigma^y + \sigma^z)$ can be rotated to $\exp(i\theta\sigma^y)$:

$$C_{\sigma_i^x}^{-\pi/8} \circ \exp\left[i\theta(\sigma^y + \sigma^z)/\sqrt{2} \right] = \exp\left(i\theta\sigma^y \right).$$

Hence

$$C_{\sigma_i^x}^{-\pi/8} \circ \exp\left[it \left((\sigma_i^y + \sigma_i^z) B_i^{yz} + \sigma_i^x B_i^x \right) \right] =$$
$$\exp\left[it \left(\sqrt{2}\sigma_i^y B_i^{yz} + \sigma_i^x B_i^x \right) \right].$$

But this we can eliminate using the Josephson coupling, since conjugation by $\sigma_i^z \otimes \sigma_{i+1}^z$ will flip the sign of both σ_i^y and σ_i^x:

$$\exp\left[it \left(\sqrt{2}\sigma_i^y B_i^{yz} + \sigma_i^x B_i^x \right) \right] \times$$
$$\left(C_{\sigma_i^z \otimes \sigma_{i+1}^z}^{\pi/2} \circ \exp\left[it \left(\sqrt{2}\sigma_i^y B_i^{yz} + \sigma_i^x B_i^x \right) \right] \right) = I.$$

Performing this in parallel for $i = 1, 3, ...$ will suppress the system-bath interaction completely. Again, there is the implicit assumption that during the on-time of the tunneling and Josephson operations H_{SB} is negligible. Regarding computation, similar comments as in the previous subsection apply, i.e., those logical operations that commute with the decoupling pulses can be turned on simultaneously with the latter, while those that anticommute cannot. However, the cases

$$H_{SB}^{xy} = \sum_i \left(\sigma_i^x + \sigma_i^y \right) B_i^{xy} + \sigma_i^z B_i^z$$

and

$$H_{SB}^{zx} = \sum_i \left(\sigma_i^z + \sigma_i^x \right) B_i^{zx} + \sigma_i^y B_i^y$$

cannot be dealt with using this decoupling method, given the available interactions. To treat these cases we must introduce an additional short-time assumption, which may or may not be more severe than the usual BB assumption.

C. Decoupling Method for System-Bath Interaction Without Symmetry

Suppose again that $|J_{ij}| \gg |\Delta_i| \gg |B_i^x|, |B_i^y|, |B_i^z|$, but that there is no symmetry in the system-bath interaction, i.e., the case of arbitrary $H_{SB} = \sum_i \vec{\sigma}_i \cdot \vec{B}_i$. Unlike the previous cases, where there was a symmetry in H_{SB}, we

now have to resort to a small time approximation in order to expand the time evolution. Namely

$$e^{i(A+B)} = \lim_{n\to\infty} \left(e^{iA/n} e^{iB/n} \right)^n = e^{iA/n} e^{iB/n} + O(\frac{1}{n^2}).$$

To see how this is useful, assume that we leave the system-bath interaction on for a short time, so that:

$$e^{-iH_{SB}t/n} = \prod_{i=1}^{N} e^{-i\sigma_i^x B_i^x t/n} e^{-i\sigma_i^y B_i^y t/n} e^{-i\sigma_i^z B_i^z t/n} + O(\frac{1}{n^2}),$$

$$(14)$$

which can be partly decoupled using the Josephson interaction. First:

$$C_{\sigma_i^z \sigma_{i+1}^z}^{\pi/2} \circ [\prod_{\alpha=x,y,z} e^{-i\sigma_i^\alpha B_i^\alpha t/n}] =$$
$$e^{i\sigma_i^x B_i^x t/n} e^{i\sigma_i^y B_i^y t/n} e^{-i\sigma_i^z B_i^z t/n}.$$

Therefore:

$$e^{-iH_{SB}t/n} \left[C_{\sum_{i=1,3,\dots}^{N-1} \sigma_i^z \sigma_{i+1}^z}^{\pi/2} \circ e^{-iH_{SB}t/n} \right] =$$
$$e^{-i\sum_i \sigma_i^z B_i^z t/n} + O(n^{-2}).$$

The remaining term can be decoupled using the tunneling Hamiltonian:

$$e^{-i\sum_i \sigma_i^z B_i^z t/n} \left[C_{\sum_i^N \sigma_i^x}^{\pi/2} \circ e^{-i\sum_i \sigma_i^z B_i^z t/n} \right] = I + O(\frac{1}{n^2}).$$

This procedure requires pulses that are short not only on the time scale of the bath inverse frequency cutoff (for the BB procedure), but also on the time scale of the system-bath interaction [in order to justify the expansion of the exponential in Eq. (14)]. We see once more that some decoherence reduction can be performed without using more resources than is required for computation.

IX. CONCLUSIONS

By encoding two physical qubits into one logical qubit, or three physical qubits into two logical qubits, we have shown how some of the building blocks that have so far been considered indispensable in superconducting phase-qubit quantum computers, can in fact be eliminated. Moreover, by grouping the code's ancilla qubits into a single "bus qubit", we were able to further simplify the engineering constraints on the fabrication of this important class of solid-state qubits. An important additional advantage of the encoding is that it allows one to eliminate potential external sources of noise and decoherence. We have shown how to use decoupling pulse methods in order to further drastically suppress the remaining sources of errors on these encoded qubits, without using extra space resources. We believe that the approach presented here will prove to be useful in reducing design constraints as well as decoherence and noise sources in superconducting phase-qubit quantum computers.

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[56] Tunneling can actually be possible at finite bias, e.g. due to finite level width. The tunneling amplitude will however be very sensitive to fluctuations of these biases. Therefore here we only consider biases large enough to suppress tunneling.

[57] This is for the case where J_{23} is kept fixed over the periods τ and τ''. One could also adjust the strength of J_{23} (voltage on the SSET) to have, e.g., $\tau = \tau''$. Which adjustment to make will be a matter of experimental convenience.

a)

$Z_1 Z_2$

b)

$Z_2 Z_3$